On the Curves which are Self-reciprocal in a Linear Nulsystem, and their Configurations in Space.

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§1.—INTRODUCTION.

Every conic curve defines in the plane a polar system; that is, a linear involutory reciprocal correspondence between the points and the lines of the plane.

In a similar way the simplest rational curve in space, the twisted cubic, defines a reciprocal correspondence also, whereby the points and the planes in space are related to each other so that to every point x corresponds the connecting plane ξ of the osculating points of the three osculating planes drawn from x onto the cubic curve, while inversely to every plane ξ corresponds the point of intersection x of the osculating planes of the cubic curve, drawn in the 3 points of intersection of the plane ξ with the cubic curve. Hence to the points of the cubic curve correspond its osculating planes, and inversely, so that this cubic curve is self-reciprocal in this polar system in space.

Now, the plane ξ always passing through its corresponding point x, this linear involutory reciprocal correspondence is coincident; that means, corresponding elements coincide. Hence this polar system in space is a nulsystem, and ξ the nulplane of the nulpol x.

The nulsystem defined by the twisted cubic is the linear nulsystem.

But reversing the reasoning, we find that to every polar system in the plane belongs a conic curve—which indeed may be imaginary—while on the other hand, a nulsystem in space does not define a curve as self-reciprocal element, but an infinite number of curves, amongst which there are straight lines, cubics and higher algebraic curves, and transcendental curves.

With these curves, which are self-reciprocals in a linear nulsystem in space, and with the configurations of these curves, we shall deal in the following, especially with cubic curves.

A curve C will be called self-reciprocal in the linear nulsystem in space, if to every point $\mathfrak{x}$ of the curve C as nulpol, corresponds the osculating plane ξ of this point as nulplane. Then, from the properties of the nulsystem follows, that the osculating points of all the osculating planes of the curve C which pass through a given point $\mathfrak{x}$, lie upon a plane ξ, which passes through point $\mathfrak{x}$ as its nulpol, and that the osculating planes in the points of intersection of the curve C with any plane ξ intersect in a point $\mathfrak{x}$, which lies upon ξ as its nulplane.

A surface S will be called self-reciprocal in the linear nulsystem, if to every point $\mathfrak{x}$ of the surface S corresponds a tangent plane ξ of this surface S as nulplane, and inversely.

But in general the nulplane ξ is not the tangent plane of S in point $\mathfrak{x}$, but in another point $\mathfrak{y}$. Only a finite number of points $\mathfrak{z}$ of the surface S can exist, which have their own tangent plane as nulplane, and shall be called the singular points of the surface S.

Surfaces of that kind, that every tangent plane is nulplane of its own tangent point, cannot exist. For in such a surface all the tangent planes through a point $\mathfrak{x}$ outside of S would touch S in a plane ξ passing through $\mathfrak{x}$, and their tangent-points therefore lie on a plane curve of n^{th} order. But the tangent points of the planes through $\mathfrak{x}$ lie on a curve of $n(n-1)^{\text{th}}$ order, and therefore

$$n = n(n-1);$$

hence

$$n = 0, \text{ the point};$$
$$n = 2, \text{ the quadric surface}.$$

But the quadric surface defines a polar system where the plane ξ cannot pass through $\mathfrak{x}$, hence never can be a nulsystem.

Plane curves cannot be self-reciprocals in the linear nulsystem except when being straight lines.

For in a plane self-reciprocal curve to every point of the curve its plane would have to correspond as nulplane, what is impossible.

Self-reciprocal straight lines exist as a tridimensional system, the "self-conjugate rays" of the nulsystem, and are grouped in plane pencils.

Self-reciprocal conics cannot exist.

Self-reciprocal cubics form a 7-dimensional system, and are grouped into pencils also on conic surfaces, which we shall treat in the following.

Self-reciprocal quartics exist also. They are rational quartics with two stationary tangents.

§2.—*Self-reciprocal Algebraic Curves in the Nulsystem.*

Besides cubics and quartics, there exist still an infinite number of self-reciprocal curves in the linear nulsystem.

For to any point $\mathfrak{a}_1$ corresponds a nulplane α_1. Now assume a point $\mathfrak{a}_2$ on α_1, infinitely near to $\mathfrak{a}_1$. To $\mathfrak{a}_2$ corresponds the nulplane α_2 which passes through $\mathfrak{a}_1$ and $\mathfrak{a}_2$. Assume $\mathfrak{a}_3$ on α_2 infinitely near to $\mathfrak{a}_2$ and produce its nulplane α_3, etc.

Then the points $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3 \ldots . \mathfrak{a}_\infty$ have as nulplanes

$$\alpha_1,\ \alpha_2 = |\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3|,\ \alpha_3 = |\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4|, \ldots . \alpha_\infty,$$

while the lines

$$t_1 = |\mathfrak{a}_1\mathfrak{a}_2|, \quad t_2 = |\mathfrak{a}_2\mathfrak{a}_3|, \ldots . t_\infty$$

are self-conjugate rays. Hence the curve consisting of the points $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3 \ldots . \infty$, having as osculating planes their nulplanes α_1, $\alpha_2 \ldots . \alpha_\infty$, and as tangents the self-conjugate rays t_1, $t_2 \ldots . t_\infty$, must therefore be a self-reciprocal curve.

Every one of these self-reciprocal curves defines the nulsystem.

All the self-reciprocal curves of the linear nulsystem have the essential property that—

1). The osculating points of all the osculating planes drawn through any point $\mathfrak{x}$, taken at random in space, lie on a plane ξ passing through $\mathfrak{x}$.

2). The osculating planes in all the points of intersection of the curve with a plane ξ, taken at random, intersect each other in a point $\mathfrak{x}$ of this plane.

For algebraic self-reciprocal curves hold the conditions—

1). The class m of the curve, that is, the number of osculating planes through any point in space, is equal to the order n of the curve; that is, the number of points of intersection of the curve with any plane in space.

2). Stationary osculating planes cannot exist (except in stationary tangents, which represent each two stationary osculating planes), for a stationary osculating plane would have two nulpols infinitely near with each other, its two osculating points.

3). The same, cuspidal points cannot exist.

4). Simple double points and double osculating planes cannot exist.

But a combination of double point and double osculating plane can exist; that is, a double point whose two osculating planes coincide in one double osculating plane.

5). To determine the rank of the curve; that is, the order of the associate developable surface or the number of tangents intersecting a straight line g, we consider the involution of n^{th} order produced on this line g.

Let g' be the conjugate line to g in the nulsystem. Then to any point $\mathfrak{p}'$ of g' corresponds a nulplane π' which passes through g. This plane π' intersects the curve C in n points $\mathfrak{x}_i$, and the n lines $|\mathfrak{p}'\mathfrak{x}_i|$ cut g in n points of an involution.

This involution contains $2(n-1)$ double points.

In each of those $2(n-1)$ points coincide two lines $|\mathfrak{p}'\mathfrak{x}_i|$. Now these lines $|\mathfrak{p}'\mathfrak{x}_i|$ are self-conjugate rays because of intersecting two conjugate rays g and g'.

Hence, when two lines $|\mathfrak{p}'\mathfrak{x}_i|$ coincide, this line must contain two points $\mathfrak{x}_i$ of the curve, infinitely near with each other; that is, it must be a tangent.

Hence $2(n-1)$ tangents of C intersect g, or

"The rank of the curve C is $r = 2(n-1)$."

6). Number of chords through a point $\mathfrak{x}$, or number of apparent nodes of the curve.

Projecting the twisted curve C from a point $\mathfrak{p}$ upon a plane ε, we derive in this plane, as the image of curve C, a curve c of n^{th} order.

The points $\mathfrak{x}'$ and tangents $t'_{\mathfrak{x}}$ of this curve c are the projections of the points $\mathfrak{x}$ and tangents $t_{\mathfrak{x}}$ of the twisted curve C.

C being of the rank $2(n-1)$, c is of the class $2(n-1)$.

Hence if d is the number of (real) double points or nodes of C, s is the number of apparent double points or of chords onto C from $\mathfrak{p}$, the curve c has $(d+s)$ double points.

If $\mathfrak{p}$ is taken at random in space, but outside of the developable surface determined by C, no tangent of C passes through $\mathfrak{p}$, and C having no cuspidal points, c has no cusps either.

Hence it is

$$2(n-1) = n(n-1) - 2(d+s),$$

or

$$d + s = \frac{(n-1)(n-2)}{2},$$

the number of apparent and true nodes of C.

Now let

n = the *order* of the self-reciprocal curve C, or the number of points of intersection with a plane ε.
m = the *class* of C, or the number of osculating planes through a point $\mathfrak{p}$.
r = the *rank* of C, or the number of tangents intersecting a line g.
d = the number of double points of C.
s = the number of apparent double points, or of chords through a point $\mathfrak{p}$.
v = the number of biosculating planes.
u = the number of osculating lines; that is, lines of intersection of two osculating planes, in a plane ε.
t = the number of stationary tangents.
k = the order of the nodal curve of the osculating developable.
l = the class of the bitangent developable.

The number of cuspidal points and stationary osculating planes being $= 0$.
Then we have

$$\begin{array}{ll} 1). & n = m. \\ 2). & d = v. \\ 3). & k = l. \\ 4). & r = 2(n-1). \\ 5). & s + d = u + v \dfrac{(n-1)(n-2)}{2}. \end{array} \qquad \text{(I)}$$

But, between these constants of the curve C,

$$n,\ m,\ r,\ d,\ s,\ v,\ u,\ t,\ k,\ l$$

exist in general the six equations

$$\begin{array}{ll} 1). & m = r(r-1) - 2k - 3(n+t). \\ 2). & r = m(m-1) - 2(u+v). \\ 3). & n + t = 3(r-m). \\ 4). & n = r(r-1) - 2l - 3(m+t). \\ 5). & r = n(n-1) - 2(s+d). \\ 6). & m + t = 3(r-n). \end{array} \qquad \text{(II)}$$

The equations (I) and (II) combined give

a). $m = n$.
b). $r = 2(n-1)$.
c). $s + d = u + v = \dfrac{(n-1)(n-2)}{2}$,
d). $k = l = 2(n-2)(n-3)$.
e). $t = 2(n-3)$.
f). $s = u,\ d = v$.

Whence we get the genus of the self-reciprocal curve

$$p = (n-1)(n-2) - 2(d+s) = 0;$$

that is,

"*The algebraic curve, which is self-reciprocal in a linear nulsystem, is of the genus zero; that is, is a rational curve.*"

From these equations (III) we derive for the first orders of self-reciprocal curves the numerical constants:

$n = m$.	r.	$s + d = u + v$.	t.	$k = l$.
1	0			
2	2			
3	4	1	0	0
4	6	3	2	4
5	8	6	4	12
6	10	10	6	24

Hence the first self-reciprocal twisted curves are the general twisted cubic, and the rational twisted quartic with two stationary tangents.

§3.—*Self-reciprocal Straight Lines and their Configurations in the Nulsystem.*

The linear nulsystem contains a tridimensional system of self-reciprocal straight lines, which are grouped into plane pencils: the self-conjugate rays of the nulsystem.

Of their configurations the most simple is the self-reciprocal tetragon, generally called "nultetrahedron."

It consists of four self-conjugate rays—

$$|\mathfrak{a}\mathfrak{b}|, \quad |\mathfrak{b}\mathfrak{c}|, \quad |\mathfrak{c}\mathfrak{d}|, \quad |\mathfrak{d}\mathfrak{a}|,$$

which form four edges of a tetrahedron, the planes of which are the nulplanes of its vertices, while the third pair of edges is a pair of conjugate rays—

$$|\mathfrak{a}\mathfrak{c}| \text{ conjugate } |\mathfrak{b}\mathfrak{d}|.$$

Another configuration is the *nulhexagon*.

Let its vertices be $\mathfrak{p}_1, \ \mathfrak{p}_2, \ldots . \mathfrak{p}_6,$

its edges, $g_{12}, \ g_{23}, \ldots . g_{61},$

its sides, $\varepsilon_{123}, \ \varepsilon_{234}, \ldots . \varepsilon_{612}.$

Then the planes $\varepsilon_{i\kappa}$ are the nulplanes of the points $\mathfrak{p}_\kappa$.

The diagonals

$$l_{14} = |\mathfrak{p}_1\mathfrak{p}_4|, \; l_{25} = |\mathfrak{p}_2\mathfrak{p}_5|, \; l_{36} = |\mathfrak{p}_3\mathfrak{p}_6|$$

are conjugate rays to the three lines of intersection of the diametrically opposite planes

$$l'_{14} = |\varepsilon_{612}\varepsilon_{345}|, \; l'_{25} = |\varepsilon_{123}\varepsilon_{456}|, \; l'_{36} = |\varepsilon_{234}\varepsilon_{561}|.$$

These six lines, $l_{14}\, l'_{14}\, l_{25}\, l'_{25}\, l_{36}\, l'_{36}$ are cut by two self-conjugate rays g_1 and g_2.

The short diagonals of the hexagon

$$p_2 = |\mathfrak{p}_1\mathfrak{p}_3|, \; p_4 = |\mathfrak{p}_3\mathfrak{p}_5|, \; p_6 = |\mathfrak{p}_5\mathfrak{p}_1|$$

are conjugate rays to the lines

$$p'_2 = |\varepsilon_1\varepsilon_3|, \; p'_4 = |\varepsilon_3\varepsilon_5|, \; p'_6 = |\varepsilon_5\varepsilon_1|.$$

Therefore $p'_2p'_4p'_6$ intersect in a point $\mathfrak{p}$ of the plane $[p_2p_4p_6]$. The same $p'_1p'_3p'_5$ intersect in a point $\mathfrak{p}'$ of the plane $[p_1p_3p_5]$.

The *nuloctagon.*

Let its vertices be

$$\mathfrak{p}_1, \; \mathfrak{p}_2, \ldots \mathfrak{p}_8,$$

its edges, $\quad g_{12}, \; g_{23}, \ldots g_{81},$

its sides, $\quad \varepsilon_1, \; \varepsilon_2, \ldots \varepsilon_8.$

Each quadruple of 4 not adjoining edges is cut by two lines,

$$g_{12}, \; g_{34}, \; g_{56}, \; g_{78} \text{ by } l_1 \text{ and } l'_1,$$

$g_{23}, g_{45}, g_{67}, g_{81}$ by l_2 and l'_2, which are conjugate rays, and these four lines

$$l_1, \; l'_1, \; l_2, \; l'_2$$

are in hyperbolic position.

Hereby to every self-reciprocal octagon is associated a ruled hyperboloid.

The *nul n-gon.*

Any quadruple of not adjoining edges is cut by two conjugate rays, and any two of these pairs determine a ruled hyperboloid, etc.

§4.—*Self-reciprocal Cubic Curves in the Nulsystem.*

Every linear nulsystem contains a 7-dimensional system of self-reciprocal twisted cubics.

Through every point $\mathfrak{p}$ passes a 5-dimensional system of self-reciprocal cubics, which in $\mathfrak{p}$ osculate the nulplane ε of $\mathfrak{p}$. Every self-conjugate ray in ε is tangent of a 4-dimensional system of this 5-dimensional system.

Every self-conjugate ray is tangent of a 5-dimensional system of self-reciprocal cubics.

Two points $\mathfrak{p}_1$ and $\mathfrak{p}_2$, with their nulplanes ε_1 and ε_2, determine a tridimensional system.

Three points $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$, with their nulplanes ε_1, ε_2, ε_3, determine a pencil of self-reciprocal cubics.

The nulsystem contains a 5-dimensional system of cubic parabolas,
a 5-dimensional " " " circles,
a 6-dimensional " " " hyperbolic parabolas.

§5.—*Self-reciprocal Curves on given Surfaces.*

On any surface F given at random, there exists an infinite number of curves, C, which are self-reciprocals in a given linear nulsystem.

Through every point of F always passes one and only one C, except in certain "singular points" $\mathfrak{S}$ of F, through which passes an infinite number of C's. These singular points $\mathfrak{S}$ are those points which have their tangent plane as nulplane. In these points $\mathfrak{S}$ the surface F is touched by the surface F', which is reciprocal to F in the nulsystem. Hence these points $\mathfrak{S}$ exist only in a finite number.

The self-reciprocal curve C, which passes through a given point $\mathfrak{x}_1$ of F, is produced by the infinitesimal method in the following way:

Let ξ_1 be the nulplane of $\mathfrak{x}_1$. Then ξ_1 intersects F in a curve, which has as tangent in $\mathfrak{x}_1$ the tangent t_1 of C. Assume on t_1 a point $\mathfrak{x}_2$ infinitely near to $\mathfrak{x}_1$. Then the nulplane ξ_2 of $\mathfrak{x}_2$ intersects F in the next tangent t_2 of C, etc.

Let $n =$ the order of the surface F, $m =$ the class, $r =$ the rank.
Then $m =$ the order of the reciprocal F', $n =$ the class, $r =$ the rank.

The order of their curve of intersection equals the class of their double-tangent developable, $= mn$, and the points of intersection of both curves are the singular points $\mathfrak{S}$.

§6.—*Self-reciprocal Curves on Ruled and Developable Surfaces.*

All the self-reciprocal curves C on a ruled surface F intersect the generatrices of F in projective ranges of points, and the tangents of C along any generatrix g from a ruled hyperboloid of self-conjugate rays which touches the

reciprocal ruled surface F' in a straight line g', reciprocal to g, and besides g' intersects F' in a curve of $2(n-1)^{\text{th}}$ order.

For the tangents t of the self-reciprocals C intersect g, the infinitely near generatrix g_1, and the conjugate ray g'.

All the self-reciprocal curves C on a given developable surface F intersect the generatrices in projective ranges of points also. The tangents of C along a given generatrix g of F form a plane pencil, with the nulpol of the tangent plane of F in g as centre. Hence infinitely near generatrices of F are cut by the self-reciprocal curves C in perspective ranges of points, and the developable surfaces of the self-reciprocal curves C intersect each other in a curve K, which is the reciprocal curve of the developable surface F.

Twisted Cubics on Ruled and Developable Surfaces.

A developable surface, containing two given twisted cubics, is determined as the locus of the common tangent planes of both.

The class of this developable is, $m = 16$. For the two twisted cubics are projected from a point in space by cones of fourth class, which have 16 generatrices in common.

Every tangent of the one twisted cubic being intersected by four tangents of the other, four tangent planes of the other pass through the tangent line of the first; that is, the curves of third order are quadruple curves of the developable surface.

The osculating curve intersects either one of the twisted cubics in 12 points, the points of intersection of the one cubic with the osculating developable of the other.

A ruled surface, containing three given twisted cubics, $C_1 C_2 C_3$, is determined as the locus of the common secants of the three cubics.

This ruled surface is of the 54^{th} order and class. For

Assume three straight lines $l_1 l_2 l_3$ at random. All the rays g intersecting these three lines $l_1 l_2 l_3$ form a ruled hyperboloid. This intersects the cubic C_1 in 6 generatrices. Hence all the rays intersecting one cubic C_1 and two straight lines $l_2 l_3$ form a ruled surface of 6^{th} order because of intersecting a line l_1 in 6 points.

This ruled surface of 6^{th} order intersects C_2 in 18 points.

Hence the secants of two cubics $C_1 C_2$ and one line l_3 form a ruled surface of 18^{th} order because of intersecting a line l_2 in 18 points.

This ruled surface of 18^{th} order intersects C_3 in 54 points, etc.

This ruled surface of 54^{th} order, R, which is determined by the three self-reciprocal cubics $C_1 C_2 C_3$, contains $C_1 C_2 C_3$ as 9-ple curves. For through any point of C_1 pass 9 generatrices of R, the common generatrices of the two cubic cones, which produce C_2 and C_3.

Besides these three 9-ple cubics, the surface R contains 108 double rays. For all the chords of C_1 which intersect a line l lie on a ruled surface of 4^{th} order. This intersects C_2 in 12 points. Hence the chords of C_1 which intersect C_2 form a ruled surface of 12^{th} order, and this intersects C_3 in 36 points. Hence there exist 36 generatrices of R which cut C_1 twice, and therefore are double generatrices of R. The same 36 double generatrices of R intersect C_2 twice, and 36 intersect C_3 twice.

The curve of intersection of a plane with the ruled surface R is of 54^{th} order and 54^{th} class, contains nine 9-ple points and 108 double points. Hence it contains still:

$$\tfrac{1}{2}\{54\times 53 - 9\times(9\times 8) - 2\times 108 - 54\} = 972 \text{ double points.}$$

The ruled surface R contains a double curve of 972^{nd} order.

On this ruled surface R of 54^{th} order exists an infinite number of self-reciprocal curves of 27^{th} order, three of which degenerate into the cubics $C_1 C_2 C_3$. No two of these reciprocal curves intersect each other. Hence the surface contains no singular points.

If the three self-reciprocal cubics $C_1 C_2 C_3$ have one point $\mathfrak{a}$ in common, the ruled surface R breaks up into—

1). The three quadratic cones, projecting $C_1 C_2 C_3$ from $\mathfrak{a}$, counted twice.

2). The osculating plane α of $C_1 C_2 C_3$, which as nulplane of $\mathfrak{a}$ is common to $C_1 C_2 C_3$, counted twice also.

3). A ruled surface of 40^{th} order, which has $\mathfrak{a}$ as singular point. If the three self-reciprocal cubics $C_1 C_2 C_3$ have 3 points $\mathfrak{abc}$ in common, the ruled surface R consists of—

1). The 9 quadratic cones, projecting $C_1 C_2 C_3$ from $\mathfrak{abc}$, counted twice.

2). The 3 osculating planes $\alpha\beta\gamma$ in $\mathfrak{abc}$ counted twice.

3). A cubic cone K, projecting the three cubics $C_1 C_2 C_3$ from the point $\mathfrak{d} = (\alpha\beta\gamma)$, counted 6-fold.

This cubic cone K, into which the ruled surface R degenerates, has the points $\mathfrak{abc}$ as singular points, and contains an infinite number of twisted cubics into which the curves of 27^{th} order of R degenerated. Its centre $\mathfrak{p} = (\alpha\beta\gamma)$ is the nulpol of the connecting plane of its 3 singular points $\pi = [\mathfrak{abc}]$. The self-reciprocal cubics then pass through 3 given points $\mathfrak{abc}$ and osculate there the three planes $\alpha\beta\gamma$. They shall be considered more particularly in §9.

§7.—*Self-reciprocal Curves on Ruled Hyperboloids.*

On every ruled hyperboloid exists an infinite number of curves which are self-reciprocals in a linear nulsystem in space.

To find the singular points of the hyperboloid, we determine its self-conjugate generatrices. For any generatrix of the hyperboloid passing through a singular point, must be a self-conjugate ray, and the singular points of the hyperboloid are therefore the points of intersection of its self-reciprocal generatrices.

Let the rays of the one system of straight lines of the hyperboloid be called *generatrices*, the rays of the other system *directrices*.

Then to all generatrices and directrices of the ruled hyperboloid H, taken at random in space, correspond in the nulsystem the generatrices and directrices of another hyperboloid H'.

To a generatrix g of H corresponds a generatrix g' of H'. g' intersects H in two points. Through each one of these two points passes a directrix d_1, viz. d_2 of H, which, because of intersecting two conjugate rays g and g', is a self-conjugate ray, and therefore common ray of both hyperboloids H and H'.

In the same way we find two self-reciprocal generatrices g_1 and g_2, which are common rays of both hyperboloids.

Hence the hyperboloid H contains 4 self-reciprocal rays, 2 generatrices, g_1, g_2, and 2 directrices, d_1, d_2, which, lying on the conjugate hyperboloid H' also, represent its curve of intersection of 4^{th} order with H.

The 4 points of intersection,

$$(g_1d_1),\ (g_1d_2),\ (g_2d_1),\ (g_2d_2)$$

are the *singular points*, and the nulplanes of these singular points are

$$[g_1d_1],\ [g_1d_2],\ [g_2d_1],\ [g_2d_2].$$

The lines $\quad |(g_1d_1),\ (g_2d_2)|$ and $|(g_1d_2),\ (g_2d_1)|$

are conjugate rays.

Each ruled hyperboloid H contains 4 *singular points, which form a nultetrahedron.*

All the self-reciprocal curves on the hyperboloid intersect any two generatrices, viz. directrices, in projective ranges of points, and the connecting lines of corresponding points form a ruled hyperboloid.

In general, no twisted cubic exists amongst the self-reciprocal curves on a ruled hyperboloid. For:

Supposed on the hyperboloid H exists a self-reciprocal cubic C. This C has two generatrices of any ruled hyperboloid through H as tangents. Hence it has the self-conjugate generatrices g_1, g_2 as tangents, and osculates them in two singular points.

But the cubic C is determined by two points with their tangents g_1, g_2 and their osculating planes, and determines an hyperboloid with g_1, g_2 as generatrices where it lies on. Only when this hyperboloid is identical with H, what in general is not the case, H contains a twisted cubic.

If a ruled hyperboloid contains a twisted cubic, it contains only one, and never more. For:

Supposed H contains besides the self-reciprocal cubic C still another one, C'.

If C and C' have different systems of generatrices of H as chords, then 2 chords, g and d', together with C and C', make up two quartic curves of intersection of H with two quadrines, which have 8 common points of intersection Now g and d' intersect C' and C in one point, each other in one point, hence C and C' must have 5 points of intersection, what is impossible.

If, on the other hand, C and C' have the same system of generatrices of H as chords, then they make up two quartic curves with two generatrices g and g'. g intersects C', g' : C in two points, hence C and C' must have four points of intersection, which could only be the four singular points of H. But four points of intersection of two cubics are possible only in the case, §9, which is excluded here, because then the connecting lines of the common points of intersection cannot be self-conjugate rays.

Hence the hyperboloid can contain more than one cubic only, if these have less than four points of intersection, and that is, if g and g' intersect each other, or, what is the same, if H degenerates into a quadricone. Hence, *quadric surfaces which contain more than one self-reciprocal cubic are cones.*

In general, an hyperboloid H contains two pairs of self-conjugate rays which form a nultetrahedron, but contains no pair of conjugate rays.

If an hyperboloid H contains one pair of conjugate rays, g and g', all the rays of this system of generatrices are grouped in pairs of conjugate rays, while the other system of generatrices consists entirely of self-conjugate rays. Then this hyperboloid H and its reciprocal hyperboloid H' coincide, and H is called a self-reciprocal hyperboloid. It contains of self-reciprocal curves only the rays of the one system of generatrices.

What has been said here with regard to the ruled hyperboloid holds in the same way, with due specialization for its special case, the ruled paraboloid. Of special interest is that paraboloid which touches the plane at infinity in its nulpol and thereby contains it as singular point.

§8.—*The Quadratic Pencil of Self-Reciprocal Cubics.*

We have seen in §7 that a quadratic surface can contain more than one self-reciprocal twisted cubic only when it degenerates into a quadricone.

But not every quadricone, but only such a cone which contains the nulplane of its centre as tangent plane, contains reciprocal cubics.

All the self-reciprocal twisted cubics which pass through a point $\mathfrak{P}$ and there osculate its nulplane Π, form a 5-dimensional system.

All those self-reciprocal twisted cubics which in a point $\mathfrak{P}$ have the line P as tangent and osculate Π which passes through P, form a 4-dimensional system. They are produced from $\mathfrak{P}$ by a tridimensional system of quadricones, which have in their common generatrix P the common tangent plane Π.

Assume at random one of those cones K.

All the generatrices p of K are conjugate to the tangents q of a plane conic curve $\mathfrak{K}$ in Π. The points $\mathfrak{q}$ of this cone correspond to the tangent planes of K, and therefore lie upon them.

Let us assume one of these quadricones K. Construe in any one of its points $\mathfrak{x}$ the self-conjugate tangent line x which passes through a point $\mathfrak{y}$ of the conjugate conic $\mathfrak{K}$ in Π and at the same time intersects the next generatrix of K in $\mathfrak{x}_1$. Construe again the tangent x_1, intersecting $\mathfrak{K}$ in $\mathfrak{y}_1$, etc., so we get on K a self-reciprocal curve $\mathfrak{x}$, $\mathfrak{x}_1 \ldots . = C$.

The curve of intersection of the osculating developable surface of this curve C with the plane Π consists of the double ray P and of the conic $\mathfrak{K}$, is therefore of 4^{th} order. Hence this developable, being of 4^{th} order, intersects K in a curve

of 8^{th} order, which consists of the double ray P and the double curve C. That means, C is a twisted cubic, q. e. d.

From a point $\mathfrak{x}$ of this twisted cubic C on the generatrix p, it is projected by a quadric cone, which intersects Π in the conic X. This conic X touches the conic $\mathfrak{K}$, which corresponds to K in the point $\mathfrak{P}$, and the point of intersection of x, the tangent of C in $\mathfrak{x}$ with Π.

Projecting X from any other point $\mathfrak{x}'$ of the same generatrix p by a quadratic cone, this cone intersects K in p and in a twisted cubic C'.

Any tangent y' of this cubic C' is produced from $\mathfrak{x}'$ by a plane $|\mathfrak{x}'y'|$, which intersects Π in the same line x_0, as the plane $|\mathfrak{x}y|$ producing the tangent y of C from $\mathfrak{x}$. Hence y and y' both pass through the point of intersection $\mathfrak{q}$ of x_0 with the tangent plane of K along $|\mathfrak{P}\mathfrak{y}|$. But this point $\mathfrak{q}$ is nulpol of this tangent plane, because of y being self-conjugate, and therefore y' is self-conjugate; that is, all the tangents of C' are self-conjugate rays, and C' therefore a self-reciprocal curve:

"*On the quadricone* K *exists an infinite number of self-reciprocal twisted cubics, which, from their points of intersection with any generatrix p of* K *are projected upon the nulplane* Π *of the centre* $\mathfrak{P}$ *of* K *by one and the same conic X. In this way to every generatrix p of* K *corresponds a conic X in* Π."

In consequence hereof,

"*All the twisted cubics intersect the generatrices of the quadricone* K *in perspective ranges of points, and their tangents produce projective pencils of rays in the tangent plane of* K."

For the lines $|\mathfrak{x}\mathfrak{y}|$, $\mathfrak{x}$ being point of intersection with the one, $\mathfrak{y}$ with the other generatrix, pass through one and the same point of Π, the point of intersection of the projection-conics X and Y. All the tangents of C along a generatrix p intersect in the nulpol of π on Π.

Any two self-reciprocal twisted cubics of the quadratic pencil have, besides the centre $\mathfrak{P}$, no point of intersection. For, if intersecting in a point $\mathfrak{z}$, they are produced from this point $\mathfrak{z}$ by the same quadratic cone, hence they are identical.

Therefore the quadratic cone K contains only one singular point, its centre $\mathfrak{P}$, which really consists of two singular points, infinitely near together, on the generatrix P.

The reciprocal surface of cone K is conic $\mathfrak{K}$.

Assume at random two cubics C_1 and C_2 of the 4-dimensional system,

$$\mathfrak{P}P\Pi.$$

They are produced from $\mathfrak{P}$ by two quadricones K_1 and K_2, which have a common generatrix P with common tangent plane Π, and contain each a quadratic pencil of self-reciprocal cubics.

These quadricones intersect in two farther rays, d and d'.

Through a point $\mathfrak{d}$ of d passes one curve of pencil K_1 and one curve of pencil K_2. These curves intersect d' in $\mathfrak{d}_1'$ and $\mathfrak{d}_2'$.

The osculating plane δ of point $\mathfrak{d}$ intersects P in point $\mathfrak{p}$. Then from $\mathfrak{p}$ both cubics C_1 and C_2 are projected by the same cubic cone with cuspidal-generatrix P.

§9.—*The Cubic Pencil of Self-reciprocal Cubics.*

The cubic pencil of self-reciprocal cubics is determined by three points, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, as common points of intersection, or three planes, α, β, γ, as common planes of osculation.

Let $\pi = [\mathfrak{a}, \mathfrak{b}, \mathfrak{c}]$ and $\mathfrak{p} = (\alpha, \beta, \gamma)$. Then π is the nulplane of $\mathfrak{p}$ and therefore passes through $\mathfrak{p}$.

All the curves C of the cubic pencil are produced from $\mathfrak{p}$ by a cubic cone K with double generatrix d and the planes α, β, γ as inflexion-planes, and the osculating planes of all the cubics C envelop a curve $\mathfrak{C}$ of third class in π with a double tangent d', which, by the nulsystem, corresponds to the double generatrix d, while the whole curve $\mathfrak{C}$ is reciprocal to the cone K.

Proof: Any two twisted cubics which pass through $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ are produced from $\mathfrak{p}$ by two cubic cones with double generatrix, which have in common the three inflexion planes α, β, γ with their inflexion generatrices $|\mathfrak{p}\mathfrak{a}|$, $|\mathfrak{p}\mathfrak{b}|$, $|\mathfrak{p}\mathfrak{c}|$, and therefore are identical, q. e. d.

The cubic cone K has $\mathfrak{p} = (\alpha\beta\gamma)$ as centre, α, β, γ as inflexion planes with $a = |\mathfrak{p}\mathfrak{a}|$, $b = |\mathfrak{p}\mathfrak{b}|$, $c = |\mathfrak{p}\mathfrak{c}|$ as inflexion generatrices, and d as double generatrix. Hence it is of the 4^{th} class.

The plane curve $\mathfrak{C}$ of 3^{rd} class has the points $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ as cuspidal points with a, b, c as cuspidal tangents; d', the conjugate ray of d, as double tangent, and therefore is of 4^{th} order.

Every ray r which is drawn through $\mathfrak{p}$ in π contains 4 points of intersection with $\mathfrak{C}$, which are the nulpoles of the 4 tangent planes drawn through r onto K.

All the osculating planes of the cubics C of the pencil envelop $\mathfrak{C}$. Hence all the tangents of the curves C intersect $\mathfrak{C}$. All the tangents of C in the points $\mathfrak{x}$ of a generatrix x of cone K lie in a plane ξ which intersects $\mathfrak{C}$ in $\mathfrak{q}$. Hence all the tangents of C in the points $\mathfrak{x}$ of generatrix x pass through point $\mathfrak{q}$ and thereby constitute a plane pencil of rays.

The tangents of C in the two tangent planes δ_1 and δ_2 of the double generatrix d pass through the tangent points $\mathfrak{d}'_1$ and $\mathfrak{d}'_2$ of the double tangent d' of $\mathfrak{C}$.

Through every point $\mathfrak{d}$ of the double generatrix d pass two cubics C, and every plane δ' through the double tangent d' osculates two cubics C.

Through every point of the cone K passes one, and only one, cubic C, with the exception of the points of the double generatrix, through which pass two cubics C. Hence the cone K contains no singular points but $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, through which pass all the curves C of the pencil, with their singular planes α, β, γ, as common osculating planes of all the curves of the pencil.

The tangents of all the curves C constitute a system of rays of 4^{th} order and class, with $\mathfrak{abc}$ as base points and $\alpha\beta\gamma$ as base planes.

From all the points $\mathfrak{x}$ of a generatrix x of the cone K the cubics C resp. are produced by quadricones upon the plane π in one and the same conic curve X.

Proof: Producing from two points $\mathfrak{x}_1$ and $\mathfrak{x}_2$ of x the two cubics C_1 and C_2, which pass through $\mathfrak{x}_1$ and $\mathfrak{x}_2$ resp. by quadricones, these quadricones intersect plane π in two conics X_1 and X_2, which have in common the three singular points $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, the point $\mathfrak{x}'$ as point of intersection of the tangents of C in the points of x, or nulpol of ξ, and the tangent x' in $\mathfrak{x}'$ as conjugate ray to x. Hence these two conics X_1 and X_2 are identically the same.

Therefore all the ∞^2 quadricones passing through the twisted cubics C of the cubic pencil intersect π in a conic *pencil* of second order—that is, through every point of π pass 2 of these conics—and fourth class, that is, every straight line in π touches 4 of these conics (because from every point of π passes one chord through any C, but through every line of π four tangents on any C), and this pencil has as envelop the curve $\mathfrak{C}$ of 4^{th} order and 3^{rd} class, with one double tangent d' and three cuspidal points $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$.

This conic pencil is projective to the rays of the cubic cone K, and to every generatrix of K corresponds a tangent plane ξ, a tangent point $\mathfrak{x}'$ on curve $\mathfrak{C}$, a tangent $t'_{\mathfrak{x}}$ of $\mathfrak{C}$ and a conic X touching $\mathfrak{C}$ in $\mathfrak{x}'$.

Assume two rays x and y of the cubic cone K and connect their correspond-

ing points of intersection with the cubics C. These connecting lines $|\mathfrak{x}\mathfrak{y}|$ are common generatrices of those quadricones which produce C from the points of x, and those which produce C from the points of y. Hence they must intersect both projection conics X and Y in π, hence pass through one point. That is,

"*The connecting lines of the corresponding points of intersection of two generatrices of* K *with the cubics C form a linear pencil of rays with its centre in π,*" and

"*All the twisted cubics C of the cubic pencil intersect the rays of the cubic cone* K *in perspective ranges of points, which have their centres of perspectivity in π.*"

"*All the cubics C of the cubic pencil can be produced as the (partial) curves of intersection of two quadricones which have the points of intersection of a pencil of rays, with centre in π, with two lines x and y as centres, and produce two conics X and Y of the nulplane π of $\mathfrak{p} = (xy)$.*"

The cubic pencil is a configuration dual to itself, and so by inverting points and planes, we derive by the laws of duality the properties—

"*All the osculating planes of C which pass through a given tangent t' of $\mathfrak{C}$ are cut by the other osculating planes of C in conics, which from $\mathfrak{p}$ are projected by one and the same quadricone. This quadricone has the planes α, β, γ as tangent planes, and touches the cubic cone* K *in that generatrix t, which corresponds in the nulsystem to tangent t' of $\mathfrak{C}$.*"

"*From any two tangents t'_1 and t'_2* of $\mathfrak{C}$ *can be produced two osculating planes ξ_1 and ξ_2 onto each cubic C. These pairs of osculating planes ξ_1 and ξ_2 intersect each other in the rays of a plane pencil, which has the point of intersection of the tangents t'_1 and t'_2 as centre, and lies in the common tangent plane of the two quadricones with centres in $\mathfrak{p}$; which correspond to the tangents t'_1 and t'_2.*"

"On the double generatrix d of K exist two coincident projective ranges of points of intersection with the cubics C, which ranges of points have only one self-corresponding point $\mathfrak{p}$."

"The double tangent d' of $\mathfrak{C}$ is the centre of two coincident projective pencils of osculating planes of the cubics C, which pencil has only one self-corresponding plane π."

"Amongst the cubics C of the pencil exists one special cubic, which breaks up into three straight lines a, b, c."

"The cubics C are produced from the tangents t' of $\mathfrak{C}$ by perspective pencils of osculating planes," etc.

To every cubic C_1 of the cubic pencil can be found two other cubics C_2 and

C_3 of the pencil which intersect C_1, besides in the points $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, still in a fourth point $\mathfrak{p}$ and $\mathfrak{p}'$ resp., on the double generatrix d. For C_1 intersects d in two points $\mathfrak{p}$ and $\mathfrak{p}'$, and through either one of these points passes another cubic C_2 and C_3 resp.

Let C_1 and C_2 be two cubics of the pencil $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ which have one further point $\mathfrak{d}$ in common. Let δ be the nulplane of $\mathfrak{d}$.

Then we can consider C_1 and C_2 as two cubics of the pencil

$(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = |\alpha, \beta, \gamma|$, with the centre $\mathfrak{p} = (\alpha, \beta, \gamma)$ and the double ray $d = |\mathfrak{p}\mathfrak{d}|$,
$(\mathfrak{a}, \mathfrak{b}, \mathfrak{d}) = |\alpha, \beta, \delta|$, " " " $\mathfrak{p}' = (\alpha, \beta, \delta)$ " " $d' = |\mathfrak{p}'\mathfrak{c}|$,
$(\mathfrak{a}, \mathfrak{d}, \mathfrak{c}) = |\alpha, \delta, \gamma|$, " " " $\mathfrak{p}'' = (\alpha, \delta, \gamma)$ " " $d'' = |\mathfrak{p}'\mathfrak{b}|$,
$(\mathfrak{d}, \mathfrak{b}, \mathfrak{c}) = |\delta, \beta, \gamma|$, " " " $\mathfrak{p}''' = (\delta, \beta, \gamma)$ " " $d''' = |\mathfrak{p}''''\mathfrak{a}|$.

Hence the points $\mathfrak{abcd}$ are interchangeable, and 2 cubics, which have 4 common points of intersection, lie in 4 cubic pencils; that is, on 4 cubic cones. Hence two such cubics are *quadruply perspective.*

"*To every cubic C_1 of a cubic pencil can be found two other cubics C_2 and C_3 of the same cubic pencil, which are quadruply perspective to C_1.*"

"*Two cubics C_1 and C_2 of a nulsystem, which have* 4 *common points of intersection $\mathfrak{abcd}$, are quadruply perspective, and the* 4 *centres of perspectivity are the* 4 *vertices of that tetrahedron $\mathfrak{p}\mathfrak{p}'\mathfrak{p}''\mathfrak{p}'''$, which, in the nulsystem, is conjugate to the tetrahedron $\mathfrak{abcd}$. The osculating planes of these two cubics can be put in correspondence with each other in four ways, so that corresponding osculating planes intersect each other in rays of a plane. These* 4 *planes, which give with the* 2 *cubics the same trace, are the sides of the tetrahedron of common points of intersection—*

$$\pi = [\mathfrak{abc}], \ \pi' = [\mathfrak{abd}], \ \pi'' = [\mathfrak{adc}], \ \pi''' = [\mathfrak{dbc}],$$

and pass through $\mathfrak{p}\mathfrak{p}'\mathfrak{p}''\mathfrak{p}'''$ resp."

"*To any cubic C exist in the nulsystem ∞^3 other cubics, which are quadruply perspective to C.*"

But these four centres of perspectivity of two quadruply perspective cubics can never be all real points.

For, as known, if the three osculating planes α, β, γ of a cubic C, which can be produced through a point $\mathfrak{p}$, are real, the chord d of C through $\mathfrak{p}$ intersects C in two imaginary points $\mathfrak{d}_i$ and $\mathfrak{d}_i'$, and if the chord d of a cubic C through a point $\mathfrak{p}$ intersects C in two real points $\mathfrak{d}$ and $\mathfrak{d}'$, two of the three osculating planes of C through $\mathfrak{p}$ are imaginary, β_i, γ_i.

Hence of the 4 points of intersection, $\mathfrak{abcd}$, and their osculating planes, $\alpha\beta\gamma\delta$, in the first case, $\mathfrak{d}_i$ and δ_i are imaginary;
in the latter case, $\mathfrak{b}_i\mathfrak{c}_i$ and $\beta_i\gamma_i$ are imaginary;
hence, of the points $\mathfrak{p}\mathfrak{p}'\mathfrak{p}''\mathfrak{p}'''$ and the planes $\pi\pi'\pi''\pi'''$
in the first case, $\mathfrak{p}_i'\mathfrak{p}_i''\mathfrak{p}_i'''$ and $\pi_i'\pi_i''\pi_i'''$ are imaginary;
in the latter case, $\mathfrak{p}_i'\mathfrak{p}_i''$ and $\pi_i'\pi_i''$ are imaginary.

That means, of the 4 centres of perspectivity of 2 quadruply perspective cubics either two are imaginary and two real, the cubics contain two real and two imaginary points of intersection, give the same trace on two real and two imaginary planes, and have two real and two imaginary common osculating planes, or

Three centres of perspectivity are imaginary and only one real, the cubics contain three real and one imaginary point of intersection, give the same trace on one real and three imaginary planes, and have three real and one imaginary osculating planes in common. Hence,

The highest number of real points of intersection or of common osculating planes of cubics of a nulsystem is two, and two cubics of a nulsystem can never be more than double perspective with two real centres of perspectivity.

Herefrom we derive the result—

"*If two cubics of the same linear nulsystem have* 4 *points,* $\mathfrak{abcd}$ *in common, they have* 4 *common osculating planes also,* $\alpha\beta\gamma\delta$, *and viz. the tetrahedron of the four common points of intersection,* T, *and the tetrahedron of the four common osculating planes,* T', *are conjugate tetrahedra of the nulsystem. The* 4 *planes of the tetrahedron of the common points of intersection* T *are cut by the osculating planes of the two cubics in the tangents of the same quartic curve of* 3^{rd} *class, and the points of both cubics are projected from the* 4 *vertices of the tetrahedron of common osculating planes,* T', *by the same cubic cones of* 4^{th} *class.*

"*Of the common points of intersection, and of the common osculating planes, not more than three and not less than two are real; of the tracing curves in the planes of the tetrahedron* T, *and of the projecting cones from the vertices of tetrahedra* T', *or centres of perspectivity of both cubics, not more than two and not less than one are real—the others imaginary.*"

The general cubic pencil contains 0, 2 or 4 cubic hyperbolic parabolas, according to the number of real points of intersection of $\mathfrak{C}$ with the line of infinity, but no cubic parabola, except if the cubic cone K passes through the

pole $\mathfrak{p}_\infty$ of the plane at infinity, π_∞. In this case $\mathfrak{C}$ is a parabolic curve; that is, has the line of infinity as tangent, and on the ray $|\mathfrak{p}\mathfrak{p}_\infty|$ the tangents of all the cubics C are parallel.

§10.—*Special Self-reciprocal Cubics and their Configurations.*

I.—Particular Positions and Coincidence of Base Points.

1). If two of three base points $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ of the cubic pencil lie upon a self-conjugate ray, $s = |\mathfrak{b}\mathfrak{c}|$, their osculating planes β and γ pass through the same ray, $s = |\beta\gamma|$, the centre of the cubic cone K is the point of intersection of s with α, the osculating plane of $\mathfrak{a}$, $\mathfrak{p} = (s\alpha)$, and the plane of the curve $\mathfrak{C}$ is $\pi = |s\mathfrak{a}|$. The cubics C, because of having 4 points of intersection with the planes β and γ, the triple point $\mathfrak{b}$ and $\mathfrak{c}$ with β, and the triple point $\mathfrak{c}$ and $\mathfrak{b}$ with γ, must break up into the double line s and the pencil of self-conjugate rays in α. The cubic cone K consists of the three planes $\alpha\beta\gamma$ and the curve $\mathfrak{C}$ of the four lines, $|\mathfrak{a}\mathfrak{b}|$, $|\mathfrak{b}\mathfrak{c}|$, $|\mathfrak{c}\mathfrak{a}|$, $|\mathfrak{a}\mathfrak{p}|$.

2). If two of the three base points $\mathfrak{b}$ and $\mathfrak{c}$ coincide into one point, the connecting line of these two coincident points, $s = |\mathfrak{b}\mathfrak{c}|$, is either self-conjugate ray or not.

If $s = |\mathfrak{b}\mathfrak{c}|$ is no self-conjugate ray, the cubic pencil can contain no real cubics, because every cubic of the pencil must pass through $\mathfrak{b} = \mathfrak{c}$ without having $s = |\mathfrak{b}\mathfrak{c}|$ as tangent. Hence, if s' is the conjugate ray of s, $\mathfrak{p}$ is the point of intersection of s' with α, $\mathfrak{p} = (s'\alpha)$, and lies upon $\pi = [\mathfrak{a}s]$. The cubic cone K consists of the double plane $[s'\mathfrak{b}]$ and the plane α; the curve $\mathfrak{C}$ of $|\mathfrak{a}\mathfrak{p}|$, $s = |\mathfrak{b}\mathfrak{c}|$ and the double line $|\mathfrak{p}\mathfrak{b}| = |\mathfrak{p}\mathfrak{c}|$. The cubics C break up into the self-conjugate rays of pencil $(\mathfrak{a}\alpha)$ and the double rays of pencil $|\mathfrak{b}s'| = |\mathfrak{c}s'|$.

If $s = |\mathfrak{b}\mathfrak{c}|$ is a self-conjugate ray of the nulsystem, the pencil consists of those twisted cubics which pass through $\mathfrak{a}$ and $\mathfrak{s} = \mathfrak{b} = \mathfrak{c}$, and have in $\mathfrak{s}$ the common tangent s. The centre of the cone K is the point of intersection $\mathfrak{p} = (s\alpha)$, is of the 3rd order and 3rd class, with s as cuspidal generatrix and α as inflexion plane. The curve $\mathfrak{C}$ lies in the plane $\pi = [\mathfrak{a}s]$, is of 3rd order and 3rd class, with s as inflexion tangent and $\mathfrak{a}$ as cuspidal point.

3). If all the base points $\mathfrak{a}\mathfrak{b}\mathfrak{c}$ coincide into one point, this point $\mathfrak{p}$ is the centre of the projection cone K, all the cubics C osculate each other in $\mathfrak{p}$, and the cone K is a quadricone while the curve $\mathfrak{C}$ in π is a conic curve. The cubics

C of this pencil have, besides the triple point $\mathfrak{p}$, no further points in common, and the pencil is the quadratic pencil treated in §8, which in this way appears as a special case of the general cubic pencil.

II.—Cylinder Pencils.

4). If the centre $\mathfrak{p}$ of the cone K lies in the plane at infinity, the cone K becomes a cylinder of 3^rd^ order and 4^th^ class. This is the case if plane $\pi = |\mathfrak{abc}|$ passes through the pole $\mathfrak{P}_\infty$ of the plane at infinity.

Such cylinder pencils exist ∞^8.

5). The most interesting of them are those which have the pole of the plane at infinity, $\mathfrak{P}_\infty$ as centre.

Their curve $\mathfrak{C}$ lies entirely in the infinity. Hence all the pencils of tangents and of connecting lines of the points of intersection of the cubics C with two generatrices x and y are parallel systems of lines. Therefore,

"All the cubics of a pencil with $\mathfrak{P}_\infty$ as centre are congruent and parallel."

"To any cubic of the nulsystem exists an infinite number of congruent and parallel cubics which lie upon a cylinder with $\mathfrak{P}_\infty$ as centre."

For every cubic is produced from $\mathfrak{P}_\infty$ by a cubic cylinder.

Such pencils exist ∞^6.

Their singular points $\mathfrak{abc}$ lie in the plane at infinity.

Hence

"*Every self-reciprocal curve of the nulsystem can be transferred parallel with itself in the direction to and fro the pole of the plane at infinity without ceasing to be self-reciprocal.*"

6). Amongst the cylinder pencils with $\mathfrak{P}_\infty$ as centre exist ∞^4 parabolic pencils, which consist of an infinite number of congruent and parallel cubic parabolas on a quadratic parabolic cylinder.

Hence the nulsystem contains ∞^5 self-reciprocal cubic parabolas.

7). Every point at infinity determines a quadratic cylindric pencil.

§11.—*The Quartic Pencil.*

I.—On the Cubic Cone.

Produce a cubic cone K with double generatrix, with a point $\mathfrak{p}$ as centre, which has the nulplane π of $\mathfrak{p}$ as tangent plane, with p as tangent generatrix.

Then this plane π intersects cone K in one further generatrix a. Let this generatrix a be inflexion-generatrix of K, with plane α as its inflexion-plane, and point $\mathfrak{a}$ as inflexion-point, a is self-conjugate ray, and α passes through $\mathfrak{p}$.

Then points $\mathfrak{p}$ and $\mathfrak{a}$ are singular points of the cone K, and therefore common points of intersection of all its self-reciprocal curves.

The two other inflexion-generatrices b and c lie in a plane ϕ with a, and the nulplanes of b and c, β and γ resp., form two plane pencils with b' and c' as axes. b' and c' lie in π, but none of the planes β and γ passes through $\mathfrak{p}$; hence they intersect the inflexion-planes β_1 and γ_1 of cone K length b and c in three points of a straight line, infinitely near with each other. That means, the tangents of the self-reciprocal curves on K, length b and c, are stationary tangents.

Let d be the double generatrix of K, with δ_1 and δ_2 as tangent planes.

The conjugate rays x' of the generatrices x of K lie in the plane π and envelop there a curve $\mathfrak{C}$ of 3rd class and 4th order, which passes through $\mathfrak{p}$ and $\mathfrak{a}$, and has in $\mathfrak{p}$ the tangent p. This curve has the points $\mathfrak{a}\mathfrak{b}'\mathfrak{c}'$ as cuspidal point, the rays $ab'c'$ as cuspidal tangents, and d', the conjugate ray of d, as double tangent with $\mathfrak{d}_1$ and $\mathfrak{d}_2$ as tangent points.

Through every point $\mathfrak{x}$ of K passes one, and only one, self-reciprocal curve which has the connecting line x_1 of $\mathfrak{x}$ with the point of intersection $\mathfrak{x}'$ of ξ and $\mathfrak{C}$, the nulpol of ξ, as tangent. In the infinitely near tangent plane ξ_1 the point of the self-reciprocal curve is derived again by its tangent x_2 intersecting $\mathfrak{x}_1$ and $\mathfrak{C}$, etc., and so the self-reciprocal curve C is produced by a line which, lying in a tangent plane and intersecting curve $\mathfrak{C}$ in the nulpol of this tangent plane, describes a developable surface.

This curve C intersects plane π in $\mathfrak{a}$ and the triple point $\mathfrak{p}$, where π osculates C. Any plane ε through $\mathfrak{p}$ intersects K in three rays x, y, z, and each of these contains a point $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ of C, while $\mathfrak{p}$ is the fourth point of intersection of ε with C.

Hence C is of the 4th order.

From any point $\mathfrak{q}$ of π pass three tangents $x'y'z'$ onto $\mathfrak{C}$, which each determine an osculating plane of C. π is osculating plane also; hence

C is of 4th class.

The osculating developable of C intersects the osculating plane π in the curve $\mathfrak{C}$ of 4th order, and the tangent p, counting twice. Hence

C is of 6th rank.

A stationary tangent must be produced from $\mathfrak{p}$ by a stationary or inflexion plane. K contains two such planes (besides α), β and γ, Hence

C has two stationary tangents, etc.

Therefore,

"*C is the rational quartic with two stationary tangents,*" and

"*Cone* K *contains a pencil of self-reciprocal quartics which pass through* $\mathfrak{a}$ *and osculate each other in* $\mathfrak{p}$."

"*d is the one trisecant chord of C.*"

"*From all the points* $\mathfrak{x}$ *of any generatrix* ξ_1 *of* K *all the quartics are produced upon* π *by one and the same plane cubic with double points.*

Hence, given one twisted quartic C_1 on K, all the others can be produced by producing the curve of intersection of π with that cubic cone which projects C_1 from one of its points $\mathfrak{x}_1$, from all the other points of the same generatrix ξ_1."

Besides in point $\mathfrak{p}$, the double ray d intersects any self-reciprocal quartic C in two points $\mathfrak{d}_1$ and δ_2. Producing C from $\mathfrak{d}_1$, we derive a cone Δ_1 which has d as double generatrix, and in d, plane δ_2 as common tangent plane with K, while it has plane $[\mathfrak{d}_2 p]$ as other tangent plane in d. Hence, its other curve of intersection is only C.

Through any point $\mathfrak{d}$ of d pass two quartics C_1 and C_2. The one has its tangent in δ_1, the other in δ_2. They are produced from $\mathfrak{d}$ by cubic cones which have d as double generatrix with one common tangent plane $[\mathfrak{d}p]$ and as other tangent plane δ_2 and δ_1 resp.

Remark: A rational cubic cone contains a *quartic* pencil if the nulplane of its centre is tangent-plane, and, besides this, intersects in an inflexion generatrix; a rational cubic cone contains a *cubic* pencil if the nulplane of its centre intersects in three inflexion generatrices.

II.—Quartic Pencil on a Quartic Cone.

Projecting a self-reciprocal quartic C from an outside point $\mathfrak{p}$ by a rational quartic cone K, we derive on K a pencil of self-reciprocal quartics.

Perspective-cone K contains three double rays $d_1 d_2 d_3$. The nulplane π of $\mathfrak{p}$ intersects C in four singular points $\mathfrak{abcd}$, with their nulplanes $\alpha\beta\gamma\delta$ as inflexion planes, and $abcd$ as inflexion generatrices. K contains two farther inflexion planes $\tau_1\tau_2$ with inflexion generatrices $t_1 t_2$, which contain the stationary tangents of the quartics. The cone K is of 4^{th} order and of 6^{th} class, and the tangents of

all the self-reciprocal quartics intersect π in the points of a curve $\mathfrak{C}$ of 6^{th} order and 4^{th} class, with six cuspidal points $\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d}\, t_1 t_2$ and three double tangents $d_1' d_2' d_3'$.

This pencil of quartics shares the property, that all the generatrices of its perspective cone K are intersected by the quartics in perspective ranges of points, with their centre of perspectivity in π; all the tangents of C along a generatrix x form a plane pencil with centre on $\mathfrak{C}$; all the quartics C are produced from the points $\mathfrak{x}$ of a generatrix x by cubic cones which intersect π in one and the same cubic curve of 4^{th} class, etc.

The greatest number of common points of intersection of two self-reciprocal quartics or their greatest number of common osculating planes is five, four of which must lie in a plane π, the nulplane of the centre of their perspective cone of 4^{th} order K, which sends a double generatrix through the fifth common point.

§12.—*The n^{tic} Pencil.*

The self-reciprocal n^{tic} can be construed as partial curve of intersection of two rational cones of $(n-1)^{\text{th}}$ order, which have their centres on the n^{tic} C, or of two rational cones of n^{th} order.

For, the constants of the n^{tic} being given in §2, the constants of its projecting cone of $(n-1)^{\text{th}}$ order from one of its points are known, and on this cone of $(n-1)^{\text{th}}$ order K an infinite number of n^{tics} can be produced either by the infinitesimal method or by intersection with $(n-1)^{\text{tic}}$ cones with the centre in one of the generatrices of K.

K contains a pencil of self-reciprocal n^{tics}.

From an outside point $\mathfrak{p}$ a self-reciprocal n^{tic} C is produced by a rational perspective-cone K of n^{th} order and $2(n-1)^{\text{th}}$ class, with $\dfrac{(n-1)(n-2)}{2}$ double-generatrices, and $n + 2(n-3) = 3(n-2)$ inflexion planes K generatrices, n of which lie in the nulplane π of the centre $\mathfrak{p}$, the other $2(n-3)$ produce the stationary tangents of C.

This cone K contains a pencil of self-reciprocal n^{tics} C which intersect each other in n singular points $\mathfrak{a}$, $\mathfrak{b}$ .... of the plane π, the nulpoles or osculating points of the singular planes or common osculating planes α, β .... which intersect each other in $\mathfrak{p}$.

Two self-reciprocal n^{tics} never contain more than $(n+1)$ common points of intersection, with common osculating planes. n of these points always lie in a plane π, while n of the common osculating planes pass through a point $\mathfrak{p}$.

All the self-reciprocal n^{tics} C of the pencil intersect the generatrices of cone K in perspective ranges of points, with their centres of perspectivity in plane π, the nulplane of centre $\mathfrak{p}$.

The tangents of all the n^{tics} of the pencil intersect plane π in the points of a curve $\mathfrak{C}$ of $2(n-1)^{\text{th}}$ order in π and n^{th} class in π, with $3(n-2)$ cuspidal points, $\mathfrak{a}$, $\mathfrak{b}$, ... and the traces of the $2(n-3)$ stationary tangents, and with $\frac{(n-1)(n-2)}{2}$ double tangents.

All the tangents of C along a generatrix x form a plane pencil with centre on $\mathfrak{C}$.

All the n^{tics} C are projected from their points of intersection $\mathfrak{x}$ with a generatrix x by cones of $(n-1)^{\text{th}}$ order and $2(n-2)^{\text{th}}$ class, which intersect π in one and the same curve of $(n-1)^{\text{th}}$ order.

All the n^{tics} C are projected from all their points upon the plane π by the curves of $(n-1)^{\text{th}}$ order of a 1-dimensional system of $(n-1)(n-2)^{\text{th}}$ order and $2(n-1)(n-2)^{\text{th}}$ class, with the curve of $2(n-2)^{\text{th}}$ order and $(n-1)^{\text{th}}$ class $\mathfrak{C}$ as envelope.

Hence all the curves of the n^{tic} pencil can be produced as the curves of intersection of $2(n-1)^{\text{tic}}$ cones, Ξ and H, which have their centres in the points $\mathfrak{x}$ and $\mathfrak{y}$ resp. of two generatrices x and y of K, have the rays of the pencil in plane $[xy]$ with centre in π resp. in common, and project the same two curves of $(n-1)^{\text{th}}$ order and $2(n-2)^{\text{th}}$ class in π resp.

Multiple perspectivity does not exist amongst higher curves than cubics.

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